Challenges in evaluating costs of known lattice attacks

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Based on attack survey from 2019 Bernstein–Chuengsatiansup–Lange–van Vredendaal.

Why analysis is important:

- Guide attack optimization.
- Guide attack selection.
- Evaluate crypto parameters.
- Evaluate crypto designs.
- Advise users on security.

Three typical attack problems

Define $\mathcal{R} = \mathbf{Z}[x]/(x^{761} - x - 1)$;
"small" $=$ all coeffs in $\{-1, 0, 1\}$;
$w = 286$; $q = 4591$.

Attacker wants to find
small weight-w secret $a \in \mathcal{R}$.

Problem 1: Public $G \in \mathcal{R}/q$ with
$aG + e = 0$. Small secret $e \in \mathcal{R}$.

Problem 2: Public $G \in \mathcal{R}/q$ and
$aG + e$. Small secret $e \in \mathcal{R}$.

Problem 3: Public $G_1, G_2 \in \mathcal{R}/q$.
Public $aG_1 + e_1, aG_2 + e_2$.
Small secrets $e_1, e_2 \in \mathcal{R}$.

Examples of target cryptosystems

Secret key: small a; small e.

Public key reveals multiplier G and approximation $A = aG + e$.

Public key for "NTRU": $G = -e/a$, and $A = 0$.

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Systematization of naming, recognizing similarity + credits:
"NTRU" $\Rightarrow$ Quotient NTRU.
"Ring-LWE" $\Rightarrow$ Product NTRU.

Encryption for Quotient NTRU:

Input small b, small d.

Ciphertext: $B = 3Gb + d$.

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Encryption for Product NTRU:

Input encoded message M.

Randomly generate

small b, small d, small c.

Ciphertext: $B = Gb + d$

and $C = Ab + M + c$.

Encryption for Quotient NTRU:
Input small b, small d.
Ciphertext: $B = 3Gb + d$.

Encryption for Product NTRU:
Input encoded message M.
Randomly generate
small b, small d, small c.
Ciphertext: $B = Gb + d$
and $C = Ab + M + c$.

Next slides: survey of G, a, e, c, M
details and variants in NISTPQC
submissions. Source: Bernstein,
"Comparing proofs of security
for lattice-based encryption".

system	parameter set	type	set of multipliers
frodo	640	Product	$(\mathbf{Z}/32768)^{640\times 640}$
frodo	976	Product	$(\mathbf{Z}/65536)^{976\times 976}$
frodo	1344	Product	$(\mathbf{Z}/65536)^{1344\times 1344}$
kyber	512	Product	$((\mathbf{Z}/3329)[x]/(x^{256}+1))^{2\times 2}$
kyber	768	Product	$((\mathbf{Z}/3329)[x]/(x^{256}+1))^{3\times 3}$
kyber	1024	Product	$((\mathbf{Z}/3329)[x]/(x^{256}+1))^{4\times 4}$
lac	128	Product	$(\mathbf{Z}/251)[x]/(x^{512}+1)$
lac	192	Product	$(\mathbf{Z}/251)[x]/(x^{1024}+1)$
lac	256	Product	$(\mathbf{Z}/251)[x]/(x^{1024}+1)$
newhope	512	Product	$(\mathbf{Z}/12289)[x]/(x^{512}+1)$
newhope	1024	Product	$(\mathbf{Z}/12289)[x]/(x^{1024}+1)$
ntru	hps2048509	Quotient	$(\mathbf{Z}/2048)[x]/(x^{509}-1)$
ntru	hps2048677	Quotient	$(\mathbf{Z}/2048)[x]/(x^{677}-1)$
ntru	hps4096821	Quotient	$(\mathbf{Z}/4096)[x]/(x^{821}-1)$
ntru	hrss701	Quotient	$(\mathbf{Z}/8192)[x]/(x^{701}-1)$
ntrulpr	653	Product	$(\mathbf{Z}/4621)[x]/(x^{653}-x-1)$
ntrulpr	761	Product	$(\mathbf{Z}/4591)[x]/(x^{761}-x-1)$
ntrulpr	857	Product	$(\mathbf{Z}/5167)[x]/(x^{857}-x-1)$
round5n1	1	Product	$(\mathbf{Z}/4096)^{636\times 636}$
round5n1	3	Product	$(\mathbf{Z}/32768)^{876\times 876}$
round5n1	5	Product	$(\mathbf{Z}/32768)^{1217\times 1217}$
round5nd	1.0d	Product	$(\mathbf{Z}/8192)[x]/(x^{586}+\ldots+1)$
round5nd	3.0d	Product	$(\mathbf{Z}/4096)[x]/(x^{852}+\ldots+1)$
round5nd	5.0d	Product	$(\mathbf{Z}/8192)[x]/(x^{1170}+\ldots+1)$
round5nd	1.5d	Product	$(\mathbf{Z}/1024)[x]/(x^{509}-1)$
round5nd	3.5d	Product	$(\mathbf{Z}/4096)[x]/(x^{757}-1)$
round5nd	5.5d	Product	$(\mathbf{Z}/2048)[x]/(x^{947}-1)$
saber	light	Product	$((\mathbf{Z}/8192)[x]/(x^{256}+1))^{2\times 2}$
saber	main	Product	$((\mathbf{Z}/8192)[x]/(x^{256}+1))^{3\times 3}$
saber	fire	Product	$((\mathbf{Z}/8192)[x]/(x^{256}+1))^{4\times 4}$
sntrup	653	Quotient	$(\mathbf{Z}/4621)[x]/(x^{653}-x-1)$
sntrup	761	Quotient	$(\mathbf{Z}/4591)[x]/(x^{761}-x-1)$
sntrup	857	Quotient	$(\mathbf{Z}/5167)[x]/(x^{857}-x-1)$
threebears	baby	Product	$(\mathbf{Z}/(2^{3120}-2^{1560}-1))^{2\times 2}$
threebears	mama	Product	$(\mathbf{Z}/(2^{3120}-2^{1560}-1))^{3\times 3}$
threebears	papa	Product	$(\mathbf{Z}/(2^{3120}-2^{1560}-1))^{4\times 4}$

short element

$\mathbf{Z}^{640\times 8}$; $\{-12,\ldots,12\}$; Pr $1,4,17,\ldots$ (spec page 23)
$\mathbf{Z}^{976\times 8}$; $\{-10,\ldots,10\}$; Pr $1,6,29,\ldots$ (spec page 23)
$\mathbf{Z}^{1344\times 8}$; $\{-6,\ldots,6\}$; Pr $2,40,364,\ldots$ (spec page 23)
$(\mathbf{Z}[x]/(x^{256}+1))^2$; $\sum_{0\le i<4}\{-0.5,0.5\}$
$(\mathbf{Z}[x]/(x^{256}+1))^3$; $\sum_{0\le i<4}\{-0.5,0.5\}$
$(\mathbf{Z}[x]/(x^{256}+1))^4$; $\sum_{0\le i<4}\{-0.5,0.5\}$
$\mathbf{Z}[x]/(x^{512}+1)$; $\{-1,0,1\}$; Pr $1,2,1$; weight $128,128$
$\mathbf{Z}[x]/(x^{1024}+1)$; $\{-1,0,1\}$; Pr $1,6,1$; weight $128,128$
$\mathbf{Z}[x]/(x^{1024}+1)$; $\{-1,0,1\}$; Pr $1,2,1$; weight $256,256$
$\mathbf{Z}[x]/(x^{512}+1)$; $\sum_{0\le i<16}\{-0.5,0.5\}$
$\mathbf{Z}[x]/(x^{1024}+1)$; $\sum_{0\le i<16}\{-0.5,0.5\}$
$\mathbf{Z}[x]/(x^{509}-1)$; $\{-1,0,1\}$
$\mathbf{Z}[x]/(x^{677}-1)$; $\{-1,0,1\}$
$\mathbf{Z}[x]/(x^{821}-1)$; $\{-1,0,1\}$
$\mathbf{Z}[x]/(x^{701}-1)$; $\{-1,0,1\}$; key correlation ≥ 0
$\mathbf{Z}[x]/(x^{653}-x-1)$; $\{-1,0,1\}$; weight 252
$\mathbf{Z}[x]/(x^{761}-x-1)$; $\{-1,0,1\}$; weight 250
$\mathbf{Z}[x]/(x^{857}-x-1)$; $\{-1,0,1\}$; weight 281
$\mathbf{Z}^{636\times 8}$; $\{-1,0,1\}$; weight $57,57$
$\mathbf{Z}^{876\times 8}$; $\{-1,0,1\}$; weight $223,223$
$\mathbf{Z}^{1217\times 8}$; $\{-1,0,1\}$; weight $231,231$
$\mathbf{Z}[x]/(x^{586}+\ldots+1)$; $\{-1,0,1\}$; weight $91,91$
$\mathbf{Z}[x]/(x^{852}+\ldots+1)$; $\{-1,0,1\}$; weight $106,106$
$\mathbf{Z}[x]/(x^{1170}+\ldots+1)$; $\{-1,0,1\}$; weight $111,111$
$\mathbf{Z}[x]/(x^{509}-1)$; $\{-1,0,1\}$; weight $68,68$; ending 0
$\mathbf{Z}[x]/(x^{757}-1)$; $\{-1,0,1\}$; weight $121,121$; ending 0
$\mathbf{Z}[x]/(x^{947}-1)$; $\{-1,0,1\}$; weight $194,194$; ending 0
$(\mathbf{Z}[x]/(x^{256}+1))^2$; $\sum_{0\le i<10}\{-0.5,0.5\}$
$(\mathbf{Z}[x]/(x^{256}+1))^3$; $\sum_{0\le i<8}\{-0.5,0.5\}$
$(\mathbf{Z}[x]/(x^{256}+1))^4$; $\sum_{0\le i<6}\{-0.5,0.5\}$
$\mathbf{Z}[x]/(x^{653}-x-1)$; $\{-1,0,1\}$; weight 288
$\mathbf{Z}[x]/(x^{761}-x-1)$; $\{-1,0,1\}$; weight 286
$\mathbf{Z}[x]/(x^{857}-x-1)$; $\{-1,0,1\}$; weight 322
$\mathbf{Z}^2$; $\sum_{0\le i<312} 2^{10i}\{-2,-1,0,1,2\}$; Pr $1,32,62,32,1$; *
$\mathbf{Z}^3$; $\sum_{0\le i<312} 2^{10i}\{-1,0,1\}$; Pr $13,38,13$; *
$\mathbf{Z}^4$; $\sum_{0\le i<312} 2^{10i}\{-1,0,1\}$; Pr $5,22,5$; *

key offset (numerator or noise or rounding method)

$\mathbf{Z}^{640\times 8}$; $\{-12,\ldots,12\}$; Pr $1,4,17,\ldots$ (spec page 23)
$\mathbf{Z}^{976\times 8}$; $\{-10,\ldots,10\}$; Pr $1,6,29,\ldots$ (spec page 23)
$\mathbf{Z}^{1344\times 8}$; $\{-6,\ldots,6\}$; Pr $2,40,364,\ldots$ (spec page 23)
$(\mathbf{Z}[x]/(x^{256}+1))^2$; $\sum_{0\le i<4}\{-0.5,0.5\}$
$(\mathbf{Z}[x]/(x^{256}+1))^3$; $\sum_{0\le i<4}\{-0.5,0.5\}$
$(\mathbf{Z}[x]/(x^{256}+1))^4$; $\sum_{0\le i<4}\{-0.5,0.5\}$
$\mathbf{Z}[x]/(x^{512}+1)$; $\{-1,0,1\}$; Pr $1,2,1$; weight $128,128$
$\mathbf{Z}[x]/(x^{1024}+1)$; $\{-1,0,1\}$; Pr $1,6,1$; weight $128,128$
$\mathbf{Z}[x]/(x^{1024}+1)$; $\{-1,0,1\}$; Pr $1,2,1$; weight $256,256$
$\mathbf{Z}[x]/(x^{512}+1)$; $\sum_{0\le i<16}\{-0.5,0.5\}$
$\mathbf{Z}[x]/(x^{1024}+1)$; $\sum_{0\le i<16}\{-0.5,0.5\}$
$\mathbf{Z}[x]/(x^{509}-1)$; $\{-1,0,1\}$; weight $127,127$
$\mathbf{Z}[x]/(x^{677}-1)$; $\{-1,0,1\}$; weight $127,127$
$\mathbf{Z}[x]/(x^{821}-1)$; $\{-1,0,1\}$; weight $255,255$
$\mathbf{Z}[x]/(x^{701}-1)$; $\{-1,0,1\}$; key correlation ≥ 0; $\cdot(x-1)$
round $\{-2310,\ldots,2310\}$ to $3\mathbf{Z}$
round $\{-2295,\ldots,2295\}$ to $3\mathbf{Z}$
round $\{-2583,\ldots,2583\}$ to $3\mathbf{Z}$
round $\mathbf{Z}/4096$ to $8\mathbf{Z}$
round $\mathbf{Z}/32768$ to $16\mathbf{Z}$
round $\mathbf{Z}/32768$ to $8\mathbf{Z}$
round $\mathbf{Z}/8192$ to $16\mathbf{Z}$
round $\mathbf{Z}/4096$ to $8\mathbf{Z}$
round $\mathbf{Z}/8192$ to $16\mathbf{Z}$
reduce mod $x^{508}+\ldots+1$; round $\mathbf{Z}/1024$ to $8\mathbf{Z}$
reduce mod $x^{756}+\ldots+1$; round $\mathbf{Z}/4096$ to $16\mathbf{Z}$
reduce mod $x^{946}+\ldots+1$; round $\mathbf{Z}/2048$ to $8\mathbf{Z}$
round $\mathbf{Z}/8192$ to $8\mathbf{Z}$
round $\mathbf{Z}/8192$ to $8\mathbf{Z}$
round $\mathbf{Z}/8192$ to $8\mathbf{Z}$
$\mathbf{Z}[x]/(x^{653}-x-1)$; $\{-1,0,1\}$; invertible mod 3
$\mathbf{Z}[x]/(x^{761}-x-1)$; $\{-1,0,1\}$; invertible mod 3
$\mathbf{Z}[x]/(x^{857}-x-1)$; $\{-1,0,1\}$; invertible mod 3
$\mathbf{Z}^2$; $\sum_{0\le i<312} 2^{10i}\{-2,-1,0,1,2\}$; Pr $1,32,62,32,1$; *
$\mathbf{Z}^3$; $\sum_{0\le i<312} 2^{10i}\{-1,0,1\}$; Pr $13,38,13$; *
$\mathbf{Z}^4$; $\sum_{0\le i<312} 2^{10i}\{-1,0,1\}$; Pr $5,22,5$; *

ciphertext offset (noise or rounding method)

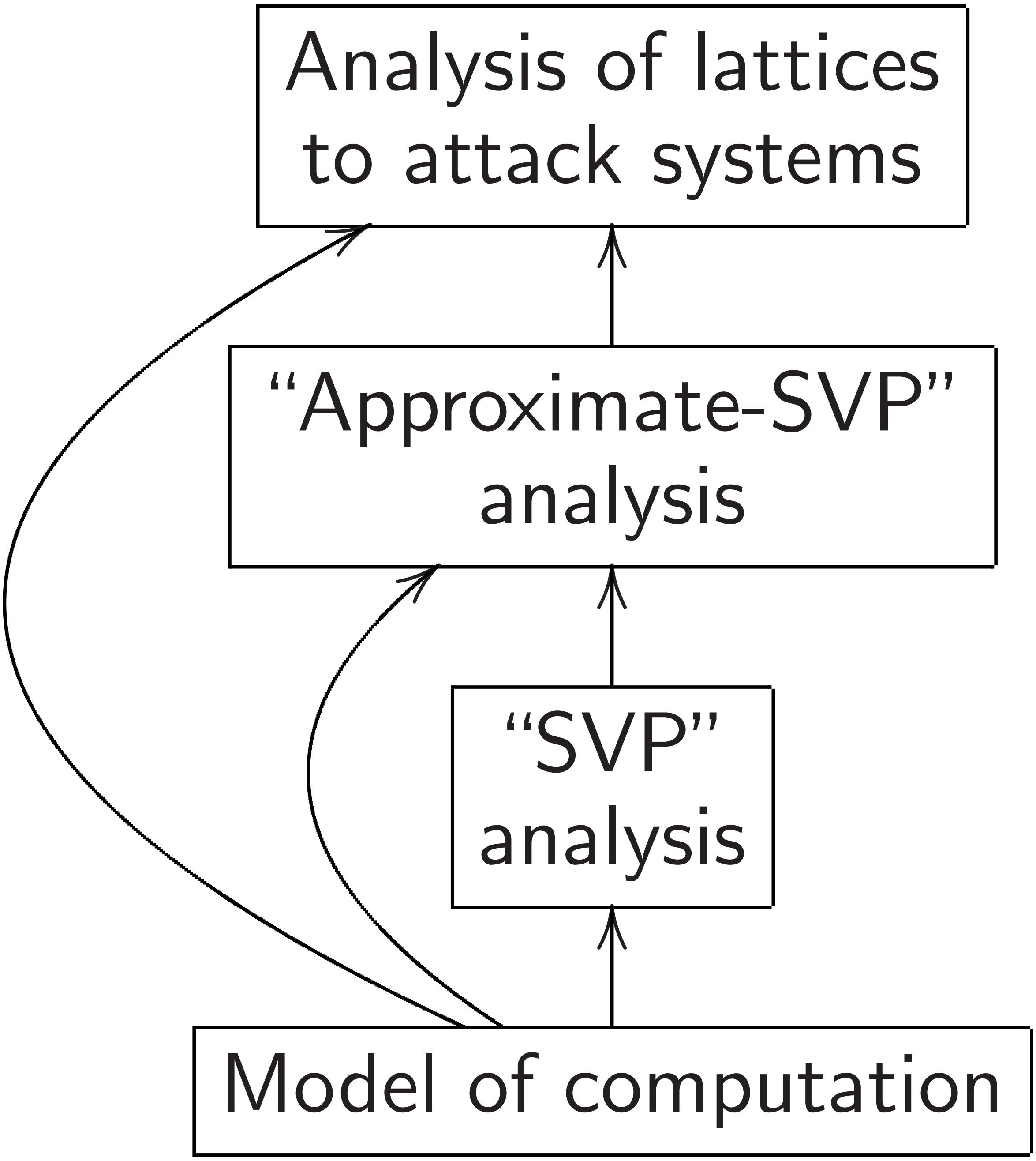
$\mathbf{Z}^{8\times 8}$; $\{-12,\dots,12\}$; Pr $1, 4, 17, \dots$ (spec page 23)
$\mathbf{Z}^{8\times 8}$; $\{-10,\dots,10\}$; Pr $1, 6, 29, \dots$ (spec page 23)
$\mathbf{Z}^{8\times 8}$; $\{-6,\dots,6\}$; Pr $2, 40, 364, \dots$ (spec page 23)
$\mathbf{Z}[x]/(x^{256}+1)$; $\sum_{0\le i<4}\{-0.5, 0.5\}$
$\mathbf{Z}[x]/(x^{256}+1)$; $\sum_{0\le i<4}\{-0.5, 0.5\}$
$\mathbf{Z}[x]/(x^{256}+1)$; $\sum_{0\le i<4}\{-0.5, 0.5\}$
$\mathbf{Z}[x]/(x^{512}+1)$; $\{-1, 0, 1\}$; Pr $1, 2, 1$
$\mathbf{Z}[x]/(x^{1024}+1)$; $\{-1, 0, 1\}$; Pr $1, 6, 1$
$\mathbf{Z}[x]/(x^{1024}+1)$; $\{-1, 0, 1\}$; Pr $1, 2, 1$
$\mathbf{Z}[x]/(x^{512}+1)$; $\sum_{0\le i<16}\{-0.5, 0.5\}$
$\mathbf{Z}[x]/(x^{1024}+1)$; $\sum_{0\le i<16}\{-0.5, 0.5\}$
not applicable
not applicable
not applicable
not applicable
bottom 256 coeffs; $z \mapsto \lfloor (114(z+2156)+16384)/32768 \rfloor$
bottom 256 coeffs; $z \mapsto \lfloor (113(z+2175)+16384)/32768 \rfloor$
bottom 256 coeffs; $z \mapsto \lfloor (101(z+2433)+16384)/32768 \rfloor$
round $\mathbf{Z}/4096$ to $64\mathbf{Z}$
round $\mathbf{Z}/32768$ to $512\mathbf{Z}$
round $\mathbf{Z}/32768$ to $64\mathbf{Z}$
bottom 128 coeffs; round $\mathbf{Z}/8192$ to $512\mathbf{Z}$
bottom 192 coeffs; round $\mathbf{Z}/4096$ to $128\mathbf{Z}$
bottom 256 coeffs; round $\mathbf{Z}/8192$ to $256\mathbf{Z}$
bottom 318 coeffs; round $\mathbf{Z}/1024$ to $64\mathbf{Z}$
bottom 410 coeffs; round $\mathbf{Z}/4096$ to $512\mathbf{Z}$
bottom 490 coeffs; round $\mathbf{Z}/2048$ to $64\mathbf{Z}$
round $\mathbf{Z}/8192$ to $1024\mathbf{Z}$
round $\mathbf{Z}/8192$ to $512\mathbf{Z}$
round $\mathbf{Z}/8192$ to $128\mathbf{Z}$
not applicable
not applicable
not applicable
$\mathbf{Z}$; $\sum_{0\le i<312} 2^{10i}\{-2, -1, 0, 1, 2\}$; Pr $1, 32, 62, 32, 1$; *
$\mathbf{Z}$; $\sum_{0\le i<312} 2^{10i}\{-1, 0, 1\}$; Pr $13, 38, 13$; *
$\mathbf{Z}$; $\sum_{0\le i<312} 2^{10i}\{-1, 0, 1\}$; Pr $5, 22, 5$; *

set of encoded messages
8×8 matrix over $\{0, 8192, 16384, 24576\}$
8×8 matrix over $\{0, 8192, \ldots, 57344\}$
8×8 matrix over $\{0, 4096, \ldots, 61440\}$
$\sum_{0 \le i < 256} \{0, 1665\} x^i$
$\sum_{0 \le i < 256} \{0, 1665\} x^i$
$\sum_{0 \le i < 256} \{0, 1665\} x^i$
256-dim subcode (see spec) of $\sum_{0 \le i < 512} \{0, 126\} x^i$
256-dim subcode (see spec) of $\sum_{0 \le i < 1024} \{0, 126\} x^i$
256-dim subcode (see spec) of $\sum_{0 \le i < 1024} \{0, 126\} x^i$
$\sum_{0 \le i < 256} \{0, 6145\} x^i (1 + x^{256})$
$\sum_{0 \le i < 256} \{0, 6145\} x^i (1 + x^{256} + x^{512} + x^{768})$
not applicable
not applicable
not applicable
not applicable
$\sum_{0 \le i < 256} \{0, 2310\} x^i$
$\sum_{0 \le i < 256} \{0, 2295\} x^i$
$\sum_{0 \le i < 256} \{0, 2583\} x^i$
8×8 matrix over $\{0, 1024, 2048, 3072\}$
8×8 matrix over $\{0, 4096, \ldots, 28672\}$
8×8 matrix over $\{0, 2048, \ldots, 30720\}$
$\sum_{0 \le i < 128} \{0, 4096\} x^i$
$\sum_{0 \le i < 192} \{0, 2048\} x^i$
$\sum_{0 \le i < 256} \{0, 4096\} x^i$
128-dim subcode (see spec) of $\sum_{0 \le i < 318} \{0, 512\} x^i$
192-dim subcode (see spec) of $\sum_{0 \le i < 410} \{0, 2048\} x^i$
256-dim subcode (see spec) of $\sum_{0 \le i < 490} \{0, 1024\} x^i$
$\sum_{0 \le i < 256} \{0, 4096\} x^i$
$\sum_{0 \le i < 256} \{0, 4096\} x^i$
$\sum_{0 \le i < 256} \{0, 4096\} x^i$
not applicable
not applicable
not applicable
256-dim subcode (see spec) of $\sum_{0 \le i < 274} \{0, 512\} 2^{10i}$
256-dim subcode (see spec) of $\sum_{0 \le i < 274} \{0, 512\} 2^{10i}$
256-dim subcode (see spec) of $\sum_{0 \le i < 274} \{0, 512\} 2^{10i}$

Attacking these problems

Attack strategy with reputation of usually being best: "primal" strategy. Focus of this talk.

Normal layers in analysis:



Models of computation

Multitape Turing machine: e.g., sort N ints, each $N^{o(1)}$ bits, in time $N^{1+o(1)}$, space $N^{1+o(1)}$.

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PRAM: multiple inequivalent definitions, untethered to physical explanations. Sort in time $N^{o(1)}$.

Quantum computing:
similar divergence of models.

Lattices

Rewrite each problem as finding **short** nonzero solution to system of homogeneous $\mathcal{R}/q$ equations.

Problem 1: Find $(a, e) \in \mathcal{R}^2$ with $aG + e = 0$, given $G \in \mathcal{R}/q$.

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Problem 2: Find $(a, t, e) \in \mathcal{R}^3$ with $aG + e = At$, given $G, A \in \mathcal{R}/q$.

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Problem 2: Find $(a, t, e) \in \mathcal{R}^3$ with $aG + e = At$, given $G, A \in \mathcal{R}/q$.

Problem 3: Find $(a, t_1, t_2, e_1, e_2) \in \mathcal{R}^5$ with $aG_1 + e_1 = A_1 t_1$, $aG_2 + e_2 = A_2 t_2$, given $G_1, A_1, G_2, A_2 \in \mathcal{R}/q$.

Recognize each solution space as a full-rank lattice:

Problem 1: Lattice is image of the map $(\overline{a}, \overline{r}) \mapsto (\overline{a}, q\overline{r} - \overline{a}G)$ from $\mathcal{R}^2$ to $\mathcal{R}^2$.

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Problem 3: Lattice is image of the map $(\overline{a}, \overline{t_1}, \overline{t_2}, \overline{r_1}, \overline{r_2}) \mapsto (\overline{a}, \overline{t_1}, \overline{t_2}, A_1\overline{t_1} + q\overline{r_1} - \overline{a}G_1, A_2\overline{t_2} + q\overline{r_2} - \overline{a}G_2)$.

Module structure

Each of these lattices is an $\mathcal{R}$-module, and thus has, generically, many independent short vectors.

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e.g. in Problem 2:
Lattice has short (a, t, e).
Lattice has short (xa, xt, xe).
Lattice has short (x^2a, x^2t, x^2e).
etc.

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Lattice has short (x^2a, x^2t, x^2e).
etc.

Many more lattice vectors are fairly short combinations of independent vectors:
e.g., $((x+1)a, (x+1)t, (x+1)e)$.

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Other problems: same speedup. e.g. Problem 2: Force many coefficients of (a, t) to be 0. Bai–Galbraith special case: Force $t = 1$, and force a few coefficients of a to be 0.

(Also slowdown if q is very large?)

Standard analysis for Problem 1

Lattice has rank $2 \cdot 761 = 1522$.

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Uniform random small weight-w secret a has length $\sqrt{w} \approx 17$.

Uniform random small secret e has length usually close to $\sqrt{1522/3} \approx 23$. (What if it's smaller? What if it's larger? Does fixed weight change security?)

Attack parameter: $k = 13$.
Force k positions in a to be 0: restrict to sublattice of rank 1509.

$\Pr[a \text{ is in sublattice}] \approx 0.2\%$.

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Pretend this analysis applies to $\mathbf{Z}[x]/(x^{761} - x - 1)$. (It doesn't.)

Write equation $e = qr - aG$
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Attack parameter: $m = 600$.

Ignore $761 - m = 161$ equations:
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Projected sublattice rank
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Attack parameter: $\lambda = 1.331876$.

Rescaling: Assign weight λ to
positions in a. Increases length
of a to $\lambda\sqrt{w} \approx 23$; increases det
to $\lambda^{748} q^{600}$. (Is this λ optimal?
Interaction with e size variation?)

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Attack parameter: $\beta = 525$.

Use BKZ-β algorithm to reduce lattice basis. (What about alternatives to BKZ?)

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(This δ formula is an *asymptotic* claim without claimed error bounds. Does not match experiments for specific d.)

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Hence the attack finds (a, e), assuming forcing worked. If it didn't, retry. (Are these tries independent? Should they use new parameters? Grover?)

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0.292β (fake) cost for "sieving" is advertised as being below $0.187\beta \log_2 \beta - 1.019\beta + 16.1$ (questionable extrapolation of experiments) for "enumeration".

Note fragility of comparison.

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Need to get analyses right!
First step: include models
that account for memory cost.

`sntrup761` evaluations from "NTRU Prime: round 2" Table 2:

Ignoring hybrid attacks:

368	185	enum, free memory cost
368	185	enum, real memory cost
153	139	sieving, free memory cost
208	208	sieving, real memory cost

Including hybrid attacks:

230	169	enum, free memory cost
277	169	enum, real memory cost
153	139	sieving, free memory cost
208	180	sieving, real memory cost

Security levels:

. . .	pre-quantum	
	. . .	post-quantum

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Search all small weight-w a.

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e.g. Problem 1: aG small so $a_1G \approx -a_2G$. (How fast are near-neighbor algorithms?)

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Now $\{(v, w, vK + wL + qr)\}$ $= \{(v, v(0, K) + zB)\}$.

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Common claim: This saves time only for sufficiently narrow $\{a\}$. (Is this true, or a calculation error in existing algorithm analyses?)