Lattice-based public-key cryptosystems

D. J. Bernstein

NIST post-quantum competition: 69 submissions in first round, from hundreds of people. (+13 submissions that NIST declared incomplete or improper.)

22 signature-system submissions. 5 lattice-based: Dilithium; DRS (broken); FALCON*; pqNTRUSign*; qTESLA.

47 encryption-system submissions.
20 lattice-based: Compact LWE* (broken); Ding*; EMBLEM; Frodo; HILA5 (CCA broken); KCL*; KINDI; Kyber; LAC; LIMA; Lizard*; LOTUS; NewHope; NTRUEncrypt; NTRU HRSS; NTRU Prime; Odd Manhattan; Round2*; SABER; Titanium.

*: submitter claims patent on this submission. Warning: even without *, submission could be covered by other patents!

First serious lattice-based encryption system: NTRU from Hoffstein–Pipher–Silverman.

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Proposed 104-byte public keys for 2^{80} security.

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NTRU paper, ANTS 1998: proposed 147-byte or 503-byte keys for 2^{77} or 2^{170} security.

Let's try NTRU on the computer.

Debian: `apt install sagemath`
Fedora: `yum install sagemath`
Source: `www.sagemath.org`
Web: `sagecell.sagemath.org`

Sage is Python 2
+ many math libraries
+ a few syntax differences:

```
sage: 10^6 # power, not xor
1000000
sage: factor(314159265358979323)
317213509 * 990371647
sage:
```

```
sage: Zx.<x> = ZZ[]
sage: # now Zx is a class
sage: # Zx objects are polys
sage: # in x with int coeffs
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x^2 + 7*x + 2
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sage: f
4*x^2 + x + 3
sage: g = Zx([2,7,1])
sage: g
x^2 + 7*x + 2
sage: f+g     # built-in add
5*x^2 + 8*x + 5
sage:
```

```
sage: f*x    # built-in mul
4*x^3 + x^2 + 3*x
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8*x^2 + 2*x + 6
sage: f*(7*x)
28*x^3 + 7*x^2 + 21*x
sage:
```

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28*x^3 + 7*x^2 + 21*x
sage: f*g
4*x^4 + 29*x^3 + 18*x^2 + 23*x
 + 6
sage:
```

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sage: f*g
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 + 6
sage: f*g == f*2+f*(7*x)+f*x^2
True
sage:
```

```
sage: # replace x^n with 1,
sage: # x^(n+1) with x, etc.
sage: def convolution(f,g):
....:     return (f*g) % (x^n-1)
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3*x^2 + 4*x + 1
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sage: convolution(f,x^2)
3*x^2 + 4*x + 1
sage: convolution(f,g)
18*x^2 + 27*x + 35
sage:
```

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....:     for j in range(n))
....:   return Zx(f)
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x^6 + x^5 + x^3 - x
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sage: randompoly()
x^6 + x^5 + x^3 - x
sage: randompoly()
-x^6 + x^5 + x^4 - x^3 - x^2 +
 x + 1
sage:
```

Will use bigger n for security.

Some choices of n
in submissions to NIST:

$n = 701$ for NTRU HRSS.
$n = 743$ for NTRUEncrypt.
$n = 761$ for `sntrup4591761`.

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Can we find better algorithms?

1998 NTRU paper took $n = 503$.

Modular reduction

For integers `u`, `q` with $\mathtt{q} > 0$, Sage's "`u%q`" always produces outputs between 0 and $\mathtt{q} - 1$.

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Warning: For polynomials $\mathtt{u}$, Sage can make the same mistake.

```
sage: def balancedmod(f,q):
sage:    g=list(((f[i]+q//2)%q)
sage:    -q//2 for i in range(n))
sage:    return Zx(g)
sage:
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sage:
sage: u = 314-159*x
sage: u % 200
-159*x + 114
sage:
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sage: u = 314-159*x
sage: u % 200
-159*x + 114
sage: (u - 400) % 200
-159*x - 86
sage:
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sage:    -q//2 for i in range(n))
sage:    return Zx(g)
sage:
sage: u = 314-159*x
sage: u % 200
-159*x + 114
sage: (u - 400) % 200
-159*x - 86
sage: balancedmod(u,200)
41*x - 86
sage:
```

```
sage: def invertmodprime(f,p):
....:     Fp = Integers(p)
....:     Fpx = Zx.change_ring(Fp)
....:     T = Fpx.quotient(x^n-1)
....:     return Zx(lift(1/T(f)))
....:
sage:
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sage: n = 7
sage: f = randompoly()
sage:
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sage: n = 7
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sage: f3 = invertmodprime(f,3)
sage:
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....:
sage: n = 7
sage: f = randompoly()
sage: f3 = invertmodprime(f,3)
sage: convolution(f,f3)
6*x^6 + 6*x^5 + 3*x^4 + 3*x^3 +
 3*x^2 + 3*x + 4
sage:
```

```
def invertmodpowerof2(f,q):
  assert q.is_power_of(2)
  g = invertmodprime(f,2)
  M = balancedmod
  C = convolution
  while True:
    r = M(C(g,f),q)
    if r == 1: return g
    g = M(C(g,2-r),q)
```

Exercise: Figure out how `invertmodpowerof2` works.

Hint: Compare `r` to previous `r`.

```
sage: n = 7
sage: q = 256
sage:
```

```
sage: n = 7
sage: q = 256
sage: f = randompoly()
sage:
```

```
sage: n = 7
sage: q = 256
sage: f = randompoly()
sage: f
-x^6 - x^4 + x^2 + x - 1
sage:
```

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sage: n = 7
sage: q = 256
sage: f = randompoly()
sage: f
-x^6 - x^4 + x^2 + x - 1
sage: g = invertmodpowerof2(f,q)
sage:
```

```
sage: n = 7
sage: q = 256
sage: f = randompoly()
sage: f
-x^6 - x^4 + x^2 + x - 1
sage: g = invertmodpowerof2(f,q)
sage: g
47*x^6 + 126*x^5 - 54*x^4 -
 87*x^3 - 36*x^2 - 58*x + 61
sage:
```

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sage: f = randompoly()
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sage: g = invertmodpowerof2(f,q)
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47*x^6 + 126*x^5 - 54*x^4 -
 87*x^3 - 36*x^2 - 58*x + 61
sage: convolution(f,g)
-256*x^5 - 256*x^4 + 256*x + 257
sage:
```

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sage: f = randompoly()
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 87*x^3 - 36*x^2 - 58*x + 61
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sage: balancedmod(_,q)
1
sage:
```

NTRU key generation

Parameters:

n, positive integer (e.g., 701);

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all coefficients in $\{-1, 0, 1\}$.

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Require d invertible mod q.
Require d invertible mod 3.

Public key: $A = 3a/d$ in the ring
$R_q = (\mathbf{Z}/q)[x]/(x^n - 1)$.

```
def keypair():
  while True:
    try:
      d = randompoly()
      d3 = invertmodprime(d,3)
      dq = invertmodpowerof2(d,q)
      break
    except:
      pass
  a = randompoly()
  publickey = balancedmod(3 *
               convolution(a,dq),q)
  secretkey = d,d3
  return publickey,secretkey
```

```
sage: A,secretkey = keypair()
sage:
```

```
sage: A,secretkey = keypair()
sage: A
-126*x^6 - 31*x^5 - 118*x^4 -
 33*x^3 + 73*x^2 - 16*x + 7
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 33*x^3 + 73*x^2 - 16*x + 7
sage: d,d3 = secretkey
sage: d
-x^6 + x^5 - x^4 + x^3 - 1
sage: convolution(d,A)
-3*x^6 + 253*x^5 + 253*x^3 -
 253*x^2 - 3*x - 3
sage:
```

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sage: convolution(d,A)
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 253*x^2 - 3*x - 3
sage: balancedmod(_,q)
-3*x^6 - 3*x^5 - 3*x^3 + 3*x^2
 - 3*x - 3
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NTRU encryption

One more parameter:
w, positive integer (e.g., 467).

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Message for encryption:
n-coeff weight-w polynomial c
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"Weight w": w nonzero coeffs,
$n - w$ zero coeffs.

Ciphertext: $C = Ab + c$ in R_q
where b is chosen randomly
from the set of messages.

```
sage: def randommessage():
....:   R = randrange
....:   assert w <= n
....:   c = n*[0]
....:   for j in range(w):
....:     while True:
....:       r = R(n)
....:       if not c[r]: break
....:     c[r] = 1-2*R(2)
....:   return Zx(c)
....:
sage: w = 5
sage: randommessage()
-x^6 - x^5 + x^4 + x^3 - x^2
sage:
```

```
sage: def encrypt(c,A):
....:     b = randommessage()
....:     Ab = convolution(A,b)
....:     C = balancedmod(Ab + c,q)
....:     return C
....:
sage:
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sage: A,secretkey = keypair()
sage: c = randommessage()
sage: C = encrypt(c,A)
sage: C
21*x^6 - 48*x^5 + 31*x^4 -
 76*x^3 - 77*x^2 + 15*x - 113
sage:
```

NTRU decryption

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Then $3ab + dc$ in R_q reveals $3ab + dc$ in $R = \mathbf{Z}[x]/(x^n - 1)$.

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Then $3ab + dc$ in R_q reveals $3ab + dc$ in $R = \mathbf{Z}[x]/(x^n - 1)$. Reduce modulo 3: dc in R_3.

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a, b, c, d have small coeffs, so $3ab + dc$ is not very big.
Assume that coeffs of $3ab + dc$ are between $-q/2$ and $q/2 - 1$.

Then $3ab + dc$ in R_q reveals $3ab + dc$ in $R = \mathbf{Z}[x]/(x^n - 1)$.
Reduce modulo 3: dc in R_3.

Multiply by $1/d$ in R_3 to recover message c in R_3.

NTRU decryption

Compute $dC = 3ab + dc$ in R_q.

a, b, c, d have small coeffs, so $3ab + dc$ is not very big. **Assume** that coeffs of $3ab + dc$ are between $-q/2$ and $q/2 - 1$.

Then $3ab + dc$ in R_q reveals $3ab + dc$ in $R = \mathbf{Z}[x]/(x^n - 1)$. Reduce modulo 3: dc in R_3.

Multiply by $1/d$ in R_3 to recover message c in R_3. Coeffs are between -1 and 1, so recover c in R.

```
sage: def decrypt(C,secretkey):
....:     M = balancedmod
....:     f,r = secretkey
....:     u=M(convolution(C,f),q)
....:     c=M(convolution(u,r),3)
....:     return c
....:
sage:
```

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....:
sage: c
x^5 + x^4 - x^3 + x + 1
sage:
```

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```

```
sage: n = 7
sage: w = 5
sage: q = 256
sage:
```

```
sage: n = 7
sage: w = 5
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sage: A,secretkey = keypair()
sage:
```

```
sage: n = 7
sage: w = 5
sage: q = 256
sage: A,secretkey = keypair()
sage: A
-101*x^6 - 76*x^5 - 90*x^4 -
 83*x^3 + 40*x^2 + 108*x - 54
sage:
```

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sage: n = 7
sage: w = 5
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 83*x^3 + 40*x^2 + 108*x - 54
sage: d,d3 = secretkey
sage:
```

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sage: w = 5
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sage: A,secretkey = keypair()
sage: A
-101*x^6 - 76*x^5 - 90*x^4 -
 83*x^3 + 40*x^2 + 108*x - 54
sage: d,d3 = secretkey
sage: d
x^5 + x^4 - x^3 + x - 1
sage:
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sage: d,d3 = secretkey
sage: d
x^5 + x^4 - x^3 + x - 1
sage: conv = convolution
sage:
```

```
sage: n = 7
sage: w = 5
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sage: A,secretkey = keypair()
sage: A
-101*x^6 - 76*x^5 - 90*x^4 -
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sage: d,d3 = secretkey
sage: d
x^5 + x^4 - x^3 + x - 1
sage: conv = convolution
sage: M = balancedmod
sage: a3 = M(conv(d,A),q)
sage:
```

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sage: n = 7
sage: w = 5
sage: q = 256
sage: A,secretkey = keypair()
sage: A
-101*x^6 - 76*x^5 - 90*x^4 -
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sage: d,d3 = secretkey
sage: d
x^5 + x^4 - x^3 + x - 1
sage: conv = convolution
sage: M = balancedmod
sage: a3 = M(conv(d,A),q)
sage: a3
3*x^2 - 3*x
```

```
sage: c = randommessage()
sage:
```

```
sage: c = randommessage()
sage: b = randommessage()
sage:
```

```
sage: c = randommessage()
sage: b = randommessage()
sage: C = M(conv(A,b)+c,q)
sage:
```

```
sage: c = randommessage()
sage: b = randommessage()
sage: C = M(conv(A,b)+c,q)
sage: C
-57*x^6 + 28*x^5 + 114*x^4 +
 72*x^3 - 37*x^2 + 16*x + 119
sage:
```

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sage: u = M(conv(C,d),q)
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sage: u = M(conv(C,d),q)
sage: u
-8*x^6 + 2*x^5 + 4*x^4 - x^3 -
 4*x^2 + 5*x + 1
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 4*x^2 + 5*x + 1
sage: conv(a3,b)+conv(c,d)
-8*x^6 + 2*x^5 + 4*x^4 - x^3 -
 4*x^2 + 5*x + 1
```

```
sage: M(u,3)
x^6 - x^5 + x^4 - x^3 - x^2 - x
 + 1
sage:
```

```
sage: M(u,3)
x^6 - x^5 + x^4 - x^3 - x^2 - x
 + 1
sage: M(conv(c,d),3)
x^6 - x^5 + x^4 - x^3 - x^2 - x
 + 1
sage:
```

```
sage: M(u,3)
x^6 - x^5 + x^4 - x^3 - x^2 - x
 + 1
sage: M(conv(c,d),3)
x^6 - x^5 + x^4 - x^3 - x^2 - x
 + 1
sage: conv(M(u,3),d3)
x^6 - x^5 - x^4 - 3*x^3 - x^2 +
 x - 3
sage:
```

```
sage: M(u,3)
x^6 - x^5 + x^4 - x^3 - x^2 - x
 + 1
sage: M(conv(c,d),3)
x^6 - x^5 + x^4 - x^3 - x^2 - x
 + 1
sage: conv(M(u,3),d3)
x^6 - x^5 - x^4 - 3*x^3 - x^2 +
 x - 3
sage: M(_,3)
x^6 - x^5 - x^4 - x^2 + x
sage:
```

```
sage: M(u,3)
x^6 - x^5 + x^4 - x^3 - x^2 - x
 + 1
sage: M(conv(c,d),3)
x^6 - x^5 + x^4 - x^3 - x^2 - x
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All coeffs of a are in $\{-1, 0, 1\}$.
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Similar comments for d, c.
Each coeff of $3ab + dc$ in R
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e.g. $w = 467$: at most 1868.
Decryption works for $q = 4096$.

What about $w = 467$, $q = 2048$?

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$a = b = c = d =$
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But coeffs are usually $<$1024
when a, d are chosen randomly.

1996 NTRU handout mentioned
no-decryption-failure option,
but recommended smaller q
with some chance of failures.
1998 NTRU paper: decryption
failure "will occur so rarely that
it can be ignored in practice".

Crypto 2003 Howgrave-Graham–Nguyen–Pointcheval–Proos–Silverman–Singer–Whyte "The impact of decryption failures on the security of NTRU encryption":

Decryption failures imply that "all the security proofs known . . . for various NTRU paddings may not be valid after all".

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Decryption failures imply that "all the security proofs known … for various NTRU paddings may not be valid after all".

Even worse: Attacker who sees some random decryption failures can figure out the secret key!

Coeff of x^{n-1} in cd is
$c_0 d_{n-1} + c_1 d_{n-2} + \ldots + c_{n-1} d_0$.

This coeff is large $\Leftrightarrow$
$c_0, c_1, \ldots, c_{n-1}$ has
high correlation with
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i.e. c is correlated with
$x^i \operatorname{rev}(d)$ for some i, where
$\operatorname{rev}(d) = d_0 + d_1 x^{n-1} + \cdots + d_{n-1} x$.

Reasonable guesses given a random decryption failure: c correlated with some $x^i \operatorname{rev}(d)$.

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Experimentally confirmed:
Average of $c \operatorname{rev}(c)$
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Eurocrypt 2002 Gentry–Szydlo algorithm then finds d.

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Attacker changes c to
$c \pm 1$, $c \pm x$, . . . , $c \pm x^{n-1}$;
$c \pm 2$, $c \pm 2x$, . . . , $c \pm 2x^{n-1}$;
$c \pm 3$, etc.

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This changes $3ab + dc$: adds
$\pm d$, $\pm xd$, $\ldots$, $\pm x^{n-1}d$;
$\pm 2d$, $\pm 2xd$, $\ldots$, $\pm 2x^{n-1}d$;
$\pm 3d$, etc.

e.g. $3ab + dc = \cdots + 390x^{478} + \cdots$,
all other coeffs in $[-389, 389]$;
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Decryption fails for big k.

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Does $3ab + dc + kxd$ also fail?
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Yes *if* $xd = \cdots + x^{478} + \cdots$,
i.e., if $d = \cdots + x^{477} + \cdots$.

Try x^2kd, x^3kd, etc.
See pattern of d coeffs.

How to handle invalid messages

Approach 1: Tell user to constantly switch keys.

For each new sender, generate new public key. Use signatures to ensure that nobody else uses key.

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If user reuses a key: Blame user for the attacks.

Approach 2: Modify encryption and decryption to eliminate invalid messages.

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Basic idea, from Crypto 1999 Fujisaki–Okamoto: After decrypting message, check whether (1) message is valid and (2) ciphertext matches reencryption of message.

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Basic idea, from Crypto 1999 Fujisaki–Okamoto: After decrypting message, check whether (1) message is valid and (2) ciphertext matches reencryption of message.

But encryption is randomized! Reencryption won't match.

Solution: Compute all randomness that was used.

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"Product NTRU" variant is not naturally deterministic.

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Can view (b, c) as message, no further randomness. "Deterministic encryption."

"Product NTRU" variant is not naturally deterministic.

Generic Fujisaki–Okamoto solution: Require sender to compute randomness as standard hash of message.

How to handle decryption failures

Eliminating invalid messages is not enough: remember attack using decryption failures for random valid messages.

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NTRU HRSS, NTRU Prime, Odd Manhattan choose q to eliminate decryption failures.

LIMA tried to eliminate decryption failures, but failed.

More claimed failure rates:

LOTUS: $<2^{-256}$.

New Hope submission: $<2^{-213}$.

KINDI: 2^{-165}.

$\vdots$

NTRUEncrypt: $<2^{-80}$.

KCL: $\approx 2^{-60}$.

Ding: $\approx 2^{-60}$, only IND-CPA.

Current debates about what decryption failure probability is small enough; whether decryption failure probabilities were calculated correctly; etc.

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If message is guessable:
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Modern "KEM-DEM" solution,
from Eurocrypt 2000 Shoup:
Choose random message.
Use hash of message as (e.g.)
AES-256-GCM key to encrypt
and authenticate user data.

Central “one-wayness” question:
Can attacker figure out
a random message given
public key and ciphertext?

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Fujisaki–Okamoto and many newer papers try to prove that all chosen-ciphertext distinguishers (“IND-CCA attacks”) are as difficult as breaking one-wayness.

Many limitations to proofs: bugs; looseness; assumptions of “ROM” or “QROM” attacks; assumptions on failure probability; etc.

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Attacker is given public key $A = 3a/d$, ciphertext $C = Ab + c$. Can attacker find c?

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Search $\binom{n}{w}2^w$ choices of b.
If $c = C - Ab$ is small: done!

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(Can this find two different messages c? Unlikely. This would also stop legitimate decryption.)

Or search 3^n choices of d. If $a = dA/3$ is small, use (a, d) to decrypt. Slightly slower but can be reused for many ciphertexts.

Equivalent keys

Secret key (a, d) is equivalent to secret key (xa, xd), secret key (x^2a, x^2d), etc.

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$n = 701$, $w = 467$:

$$\binom{n}{w}2^w \approx 2^{1106.09};$$
$$3^n \approx 2^{1111.06};$$
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Exercise: Find more equivalences!

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Exercise: Find more equivalences!

But if w is chosen smaller then $\binom{n}{w}2^w$ search will be faster.

Collision attacks

Write d as $d_1 + d_2$ where
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Enumerate all $H(-(A/3)d_2)$.
Enumerate all $H((A/3)d_1)$.
Search for collisions.
Only about $3^{n/2}$ computations;
but beware cost of memory.

Lattices

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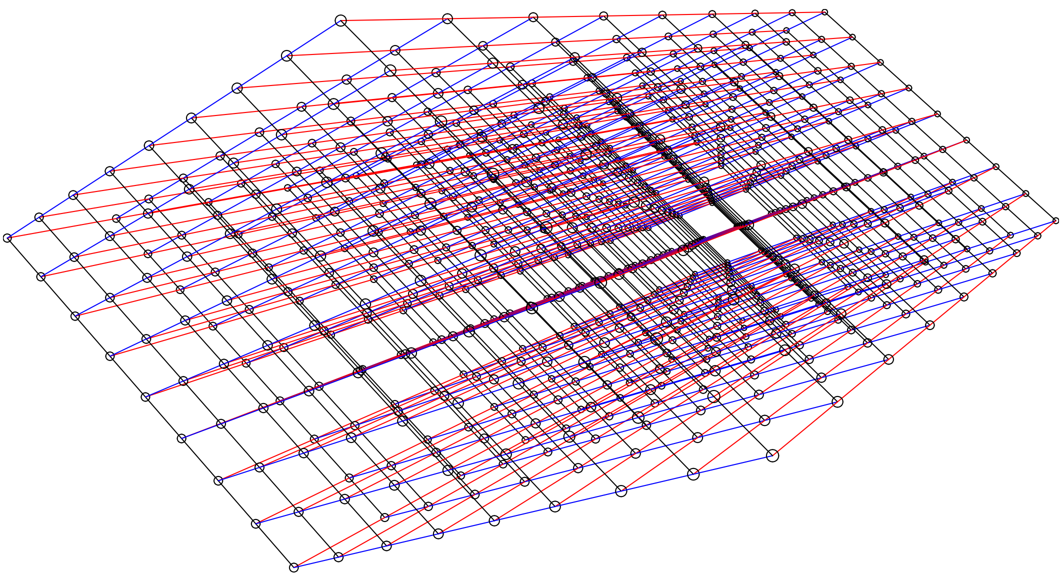

Lattices, mathematically

Assume that $b_1, b_2, \ldots, b_k \in \mathbf{R}^n$
are $\mathbf{R}$-linearly independent,
i.e., $\mathbf{R}b_1 + \ldots + \mathbf{R}b_k =$
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$b_1, \ldots, b_k$
is a **basis** of this lattice.

Short vectors in lattices

Given $b_1, b_2, \ldots, b_k \in \mathbf{Z}^n$,
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Fancier algorithms (e.g., BKZ)
compute shorter vectors
at surprisingly high speed.

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a is obtained from
$q, qx, qx^2, \ldots, qx^{n-1}$,
$A/3, xA/3, \ldots, x^{n-1}A/3$
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Write $A/3$ as

$H_0 + H_1 x + \ldots + H_{n-1} x^{n-1}$.

$(a_0, a_1, \ldots, a_{n-1}, d_0, d_1, \ldots, d_{n-1})$
is obtained from
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$(0, q, \ldots, 0, 0, 0, \ldots, 0)$,
$\vdots$
$(0, 0, \ldots, q, 0, 0, \ldots, 0)$,
$(H_0, H_1, \ldots, H_{n-1}, 1, 0, \ldots, 0)$,
$(H_{n-1}, H_0, \ldots, H_{n-2}, 0, 1, \ldots, 0)$,
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$(H_1, H_2, \ldots, H_0, 0, 0, \ldots, 1)$
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Exercise: Describe search for
(b, c) as a problem of finding
a vector close to a lattice.

Quotient NTRU vs. product NTRU

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Alice generates $A = 3a/d$ in R_q for small random a, d: i.e., $dA - 3a = 0$ in R_q.

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Alice reconstructs $3ab + dc$ in R, using smallness of a, b, d, c. Alice computes dc in R_3, deduces c, deduces b.

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where b, c, e are small and
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Alice computes $C - aB$ in R_q, i.e., $m + db + c - ae$ in R_q. Alice reconstructs m, using smallness of d, b, c, a, e.